Int J Theor Phys (2011) 50: 200–207
DOI 10.1007/s10773-010-0508-x

Contracted Hamiltonian on Symmetric Space $SU(3)/SU(2)$ and Conserved Quantities

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Received: 19 July 2010 / Accepted: 20 September 2010 / Published online: 9 October 2010

Abstract We study the quantum model on symmetric space $SU(3)/SU(2)$. By using the Inonu-Wigner contraction to Lie algebra $su(3)$, we arrive at a special case of three-body Sutherland model. It has shown that by calculating conservative quantities of this model, it has Poincare Lie algebra, too.

Keywords Inonu-Wigner contraction · Hamilton-Jacobi equation · Conserved quantity · $SU(3)/SU(2)$ symmetric space

1 Introduction

In recent years, to find solvable quantum models and integrable Hamiltonians are an important problems in theoretical and mathematical physics. Any integrable system may be constructed by reduction of a free Hamiltonian [1]. Such systems, which are not many, play a fundamental role in description of physical systems, because of their many interesting properties from both mathematical and physical points of view. Some of these Hamiltonian systems are Harmonic oscillator, Kepler problem, Morse [2], Poschl-Teller [3], Winternitz et al. [4], Evans [5–7], Calogero [8] and Sutherland [9] potentials.

One Hamiltonian system is said to be completely integrable if there exist $(N-1)$ integrals of motions and N constant of motion in involution, where one of them is the Hamiltonian [11]. When there is more than $(N-1)$ integrals of motion (not all of them in involution), the system is called superintegrable. Superintegrablity is also closely related to the fact that the Hamilton-Jacobi (H-J) equation is separable in more than one coordinate system [10]. Izmest'ev et al. [12] showed that the separable coordinates of free systems on the two-sphere can be turned into the separable coordinates on the Euclidean plane by using contractions.

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In this work, we show that the system $SU(3)/SU(2)$ lives on the two-sphere and then it is reduced to Euclidean space R^2 by Inonu-Wigner contraction. The obtained model is a three-body quantum Sutherland model with special interaction between particles and it has three constants of motion. We also show that it is a superintegrable system separated in different coordinates.

The paper is organized as follows: in Sect. 2, the Killing vectors of symmetric space $SU(3)/SU(2)$ and the Casimir operator are determined. In Sect. 3, we apply the Inonu-Wigner contraction on the obtained generators of $su(3)$ Lie algebra together with a suitable Fourier transformation. We solve the (H-J) equation of the reduced quantum model and compute its conservative quantities in Sect. 4. Finally we show that this quantum system has the Poincare Lie algebra. Some conclusions are presented in Sect. 5.

2 Killing Vectors on Symmetric Space $SU(3)/SU(2)$

According to [14], an element of the group $SU(3)$ can be written as the product of three $SU(2)$ subgroups

$$g(\alpha_1,\beta_1,\gamma_1,\alpha_2,\beta_2,\alpha_3,\beta_3,\gamma_3)=R_{23}(\alpha_1,\beta_1,\gamma_1)R_{12}(\alpha_2,\beta_2,\alpha_2)R_{23}(\gamma_3,\beta_3,\alpha_3), \tag{2.1}$$

where $R_{ij}(\alpha_k,\beta_k,\gamma_k)$ is the Euler representation $(SU(2))_{ij}$ in terms of Euler angles $(\alpha_k,\beta_k,\gamma_k)$. As an example, we have

$$R_{23}(\alpha,\beta,\gamma)=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos(\frac{\beta}{2})e^{i(\alpha+\gamma)/2} & \sin(\frac{\beta}{2})e^{-i(\alpha-\gamma)/2}\\ 0 & -\sin(\frac{\beta}{2})e^{i(\alpha-\gamma)/2} & \cos(\frac{\beta}{2})e^{-i(\alpha+\gamma)/2}\end{pmatrix}. \tag{2.2}$$

By calculating the left invariant one form $g^{-1}dg=e^i_\mu dx^\mu\lambda_i$, we get the left invariant fiel-beins e^i_μ, where λ_i is the basis of Lie algebra $su(3)$. In order to obtain the left invariant vector fields as: $L_i=e^\mu_i\frac{\partial}{\partial x^\mu}$, we should inverse the left invariant fiel-beins. Also, if we project these vector fields over a base manifold $SU(3)/SU(2)$, we can obtain Killing vectors, which are independent of the $SU(2)$ coordinates of the stability subgroup $R_{23}(\alpha_1,\beta_1,\gamma_1)$.

To this aim, it is convenient to change the variables as

$$\beta_2=2\theta,\qquad \beta_3=2\varphi,\qquad \gamma_3=2\gamma,\qquad \alpha_2=\frac{\chi_1+\chi_2}{2},\qquad \alpha_3=-\chi_1+\chi_2,$$

also by choosing the usual Gell-Mann matrices as the basis of $su(3)$ Lie algebra, we get the following realization for the Killing invariant vector fields defined over $SU(3)/SU(2)$ manifold

$$\begin{aligned}L_\pm=\frac{1}{2}e^{\pm i(\chi_1-\gamma)}\Bigg[&\pm\cos(\varphi)\frac{\partial}{\partial\theta}\mp\cot(\theta)\sin(\varphi)\frac{\partial}{\partial\varphi}-i\frac{\cot(\theta)}{2\cos(\varphi)}\frac{\partial}{\partial\gamma}\\ &+i\,\frac{\cos^2(\theta)-2\sin^2(\theta)\cos^2(\varphi)}{\sin(2\theta)\cos(\varphi)}\frac{\partial}{\partial\chi_1}-i\frac{\cos^2(\theta)+2\sin^2(\theta)\cos^2(\varphi)}{\sin(2\theta)\cos(\varphi)}\frac{\partial}{\partial\chi_2}\Bigg],\end{aligned} \tag{2.3}$$

$$\begin{aligned}Y_\pm=\frac{1}{2}e^{\pm i(\chi_1+\gamma)}\Bigg[&\pm\sin(\varphi)\frac{\partial}{\partial\theta}\pm\cot(\theta)\cos(\varphi)\frac{\partial}{\partial\varphi}+i\frac{\cot(\theta)}{2\sin(\varphi)}\frac{\partial}{\partial\gamma}\\ &+i\,\frac{\cos^2(\theta)-2\sin^2(\theta)\sin^2(\varphi)}{\sin(2\theta)\sin(\varphi)}\frac{\partial}{\partial\chi_1}-i\frac{\cos^2(\theta)+2\sin^2(\theta)\sin^2(\varphi)}{\sin(2\theta)\sin(\varphi)}\frac{\partial}{\partial\chi_2}\Bigg],\end{aligned} \tag{2.4}$$

$$X_{\pm}=\frac{1}{2}e^{\mp 2i\gamma}\left[\mp\frac{\partial}{\partial\varphi}+i\cot(2\varphi)\frac{\partial}{\partial\gamma}+\frac{i}{\sin(2\varphi)}\left(\frac{\partial}{\partial\chi_1}-\frac{\partial}{\partial\chi_2}\right)\right], \tag{2.5}$$

$$L_3=\frac{i}{4}\left(\frac{\partial}{\partial\gamma}-3\frac{\partial}{\partial\chi_1}-\frac{\partial}{\partial\chi_2}\right),\qquad L_8=\frac{i\sqrt{3}}{12}\left(-3\frac{\partial}{\partial\gamma}-3\frac{\partial}{\partial\chi_1}-\frac{\partial}{\partial\chi_2}\right), \tag{2.6}$$

where $L_{\pm}=\frac{1}{2}(L_1\pm iL_2)$, $Y_{\pm}=\frac{1}{2}(L_4\pm iL_5)$ and $X_{\pm}=\frac{1}{2}(L_6\mp iL_7)$. One can show that the above generators satisfy the following commutation relations of $su(3)$ Lie algebra

$$\begin{aligned}
&[L_3,L_{\pm}]=\pm L_{\pm},\qquad [L_3,Y_{\pm}]=\pm\frac{1}{2}Y_{\pm},\qquad [L_3,X_{\pm}]=\pm\frac{1}{2}X_{\pm},\\
&[L_8,Y_{\pm}]=\pm\frac{\sqrt{3}}{2}Y_{\pm},\qquad [L_8,X_{\pm}]=\mp\frac{\sqrt{3}}{2}X_{\pm},\\
&[L_+,L_-]=2L_3,\qquad [L_{\pm},Y_{\mp}]=\mp X_{\pm},\qquad [L_{\pm},X_{\mp}]=\pm Y_{\pm},\\
&[Y_+,Y_-]=L_3+\sqrt{3}L_8,\qquad [X_+,X_-]=L_3-\sqrt{3}L_8,\qquad [Y_{\pm},X_{\pm}]=\pm L_{\pm}.
\end{aligned}$$

Also, the $su(3)$ quadratic Casimir operator which is defined as

$$C=\frac{1}{2}(L_+L_-+L_-L_+)+\frac{1}{2}(Y_+Y_-+Y_-Y_+)+\frac{1}{2}(X_+X_-+X_-X_+)+L_3^2+L_8^2,$$

is calculated as (by ignoring the factor $\frac{1}{4}$)

$$\begin{aligned}
C=-\Bigg[&\frac{\partial^2}{\partial\theta^2}+\frac{1}{\sin^2(\theta)}\frac{\partial^2}{\partial\varphi^2}+\frac{1}{\sin^2(\theta)\sin^2(2\varphi)}\frac{\partial^2}{\partial\gamma^2}\\
&+\frac{2(4\cos^2(\theta)-1)}{\sin(2\theta)}\frac{\partial}{\partial\theta}+\frac{2\cot(2\varphi)}{\sin^2(\theta)}\frac{\partial}{\partial\varphi}+\frac{\cos^2(\theta)+\sin^2(\theta)\sin^2(2\varphi)}{\sin^2(\theta)\cos^2(\theta)\sin^2(2\varphi)}\frac{\partial^2}{\partial\chi_1^2}\\
&+\frac{3\cos^2(\theta)+3\sin^2(\theta)\sin^2(2\varphi)+4\sin^2(\theta)\cos^2(\theta)\sin^2(2\varphi)}{3\sin^2(\theta)\cos^2(\theta)\sin^2(2\varphi)}\frac{\partial^2}{\partial\chi_2^2}\\
&-2\frac{\cos^2(\theta)-\sin^2(\theta)\sin^2(2\varphi)}{\sin^2(\theta)\cos^2(\theta)\sin^2(2\varphi)}\frac{\partial^2}{\partial\chi_1\partial\chi_2}-2\frac{\cos(2\varphi)}{\sin^2(\theta)\sin^2(2\varphi)}\left(\frac{\partial^2}{\partial\gamma\partial\chi_2}-\frac{\partial^2}{\partial\gamma\partial\chi_1}\right)\Bigg].
\end{aligned} \tag{2.7}$$

Similarly we can calculate the $SU(3)$ right invariant vector fields, having minus structure constant, but its quadratic Casimir operator is the same as the left one. Also one can show that the Casimir operator is the same as Laplace-Beltrami operator of adjoint invariant metric.

3 Inonu-Wigner Contraction and the Quantum Model

According to [15], for applying the Inonu-Wigner contraction to the generators of Lie algebra $su(3)$ given by (2.3), (2.4), (2.5) and (2.6), we first change the coordinate $\theta=\frac{\rho}{R}$ and relate the new contracted generators to the old ones by $L^c_{\pm}=\frac{1}{R}L_{\pm}$, $L^c_3=L_3$, $X^c_{\pm}=X_{\pm}$, $Y^c_{\pm}=\frac{1}{R}Y_{\pm}$, $L^c_8=L_8$. Then in the limit of $R\to\infty$, the realization of $su(3)$ bases reduce to

$$L^c_{\pm}=\frac{1}{2}e^{\pm i(\chi_1-\gamma)}\left[\pm\cos(\varphi)\frac{\partial}{\partial\rho}\mp\frac{\sin(\varphi)}{\rho}\frac{\partial}{\partial\varphi}-\frac{i}{2\rho\cos(\varphi)}\left(\frac{\partial}{\partial\gamma}-\frac{\partial}{\partial\chi_1}+\frac{\partial}{\partial\chi_2}\right)\right], \tag{3.1}$$

$$Y_{\pm}^{c}=\frac{1}{2}e^{\pm i(\chi_1+\gamma)}\left[\pm\sin(\varphi)\frac{\partial}{\partial\rho}\pm\frac{\cos(\varphi)}{\rho}\frac{\partial}{\partial\varphi}+\frac{i}{2\rho\sin(\varphi)}\left(\frac{\partial}{\partial\gamma}+\frac{\partial}{\partial\chi_1}-\frac{\partial}{\partial\chi_2}\right)\right], \tag{3.2}$$

$$X_{\pm}^{c}=\frac{1}{2}e^{\mp 2i\gamma}\left[\mp\frac{\partial}{\partial\varphi}+i\cot(2\varphi)\frac{\partial}{\partial\gamma}+\frac{i}{\sin(2\varphi)}\left(\frac{\partial}{\partial\chi_1}-\frac{\partial}{\partial\chi_2}\right)\right], \tag{3.3}$$

$$L_3^c=\frac{i}{4}\left(\frac{\partial}{\partial\gamma}-3\frac{\partial}{\partial\chi_1}-\frac{\partial}{\partial\chi_2}\right),\qquad L_8^c=\frac{i\sqrt{3}}{12}\left(-3\frac{\partial}{\partial\gamma}-3\frac{\partial}{\partial\chi_1}-\frac{\partial}{\partial\chi_2}\right) \tag{3.4}$$

with the following nonvanishing commutation relations

$$[L_3^c,L_\pm^c]=\pm L_\pm^c,\qquad [L_3^c,Y_\pm^c]=\pm\frac{1}{2}Y_\pm^c,\qquad [L_3^c,X_\pm^c]=\pm\frac{1}{2}X_\pm^c,$$

$$[L_8^c,Y_\pm^c]=\pm\frac{\sqrt{3}}{2}Y_\pm^c,\qquad [L_8^c,X_\pm^c]=\mp\frac{\sqrt{3}}{2}X_\pm^c,$$

$$[L_\pm^c,X_\mp^c]=\pm Y_\pm^c,\qquad [X_+^c,X_-^c]=L_3^c-\sqrt{3}L_8^c,\qquad [Y_\pm^c,X_\pm^c]=\pm L_\pm^c.$$

It is easy to show that, the generators L_-^c, L_+^c, Y_-^c and Y_+^c commute with each other. Also the quadratic Casimir operator (2.7) reduces to (after the similarity transformation $\tilde{C}^c=f^{-1}(\rho,\varphi)C^c f(\rho,\varphi)$ with $f(\rho,\varphi)=\frac{1}{\rho\sqrt{\sin(2\varphi)}}$)

$$\begin{aligned}-\tilde{C}^c&=\frac{1}{2}(\tilde{L}_-^c\tilde{L}_+^c+\tilde{L}_+^c\tilde{L}_-^c+\tilde{Y}_-^c\tilde{Y}_+^c+\tilde{Y}_+^c\tilde{Y}_-^c)\\&=+\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}+\frac{1}{\rho^2\sin^2(2\varphi)}\left(\frac{\partial^2}{\partial\gamma^2}+\frac{\partial^2}{\partial{\chi_1}^2}+\frac{\partial^2}{\partial{\chi_2}^2}\right)+\frac{1}{\rho^2\sin^2(2\varphi)}\\&\quad+\frac{2\cos(2\varphi)}{\rho^2\sin^2(2\varphi)}\left(\frac{\partial^2}{\partial\gamma\partial\chi_1}-\frac{\partial^2}{\partial\gamma\partial\chi_2}\right)-\frac{2}{\rho^2\sin^2(2\varphi)}\frac{\partial^2}{\partial\chi_1\partial\chi_2},\end{aligned} \tag{3.5}$$

where $\tilde{Y}_\pm^c=Y_\pm^c\mp\frac{1}{4\rho\sin(\varphi)}e^{\pm i(\chi_1+\gamma)}$ and $\tilde{L}_\pm^c=L_\pm^c\mp\frac{1}{4\rho\cos(\varphi)}e^{\pm i(\chi_1-\gamma)}$. Now, by using the Fourier transformation with the kernel $\exp(i[(m_1+m_2)\chi_1+l\chi_2+(m_1-m_2)\gamma])$, (3.5) leads to following Hamiltonian

$$\begin{aligned}\tilde{H}^c(\rho,\varphi)=-\Bigg[&\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}\\&+\frac{1}{\rho^2\sin^2(2\varphi)}\left(4m_2(l-m_2)-l^2+1-4(m_1-m_2)(m_1+m_2-l)\cos^2(\varphi)\right)\Bigg].\end{aligned} \tag{3.6}$$

The above Hamiltonian interprets the motion of a particle over R^2 manifold with a scalar potential in terms of three parameters $l,\ m_1,\ m_2$. This potential is shape invariant which can be solved by algebraic method according to the concept of supersymmetry in non-relativistic quantum mechanics. In other word, the Hamiltonian can be factorized as a product of lowering and rising operators which satisfy the shape invariant symmetry. In fact, after Inuno-Wigner contraction and Fourier transformation of the Casimir eigenvalue equation, one can obtain a hierarchy of isospectral Hamiltonians labeled by the parameters m_1 and m_2. In [13], this study has been done for another parametrization of the $SU(3)$ group.

Now in this work, we show that this quantum system can be obtained from three-body Sutherland model with special interaction between particles. We also show that it has three conserved quantities which satisfy the Poincare Lie algebra and it is a superintegrable model.

4 Three Body Integrable Model and Conserved Quantities

Much of the information about any Lie symmetry are contained in properties of the group elements near the identity. By doing some mathematical calculations near the identity, for the Poincare group, one finds the following commutation relations of the Poincare Lie algebra [16]

$$\begin{aligned}&[P_\nu, P_\mu]=0, \qquad [P_\nu, J_{k\lambda}]=i(g_{\nu\lambda}P_k - g_{\nu k}P_\lambda), \qquad [P_\nu, \Theta_\mu]=0,\\ &[J_{\nu\mu}, J_{k\lambda}]=i(g_{k\mu}J_{\lambda\nu}+g_{\mu\lambda}J_{\nu k}+g_{\lambda\nu}J_{k\mu}+g_{\nu k}J_{\mu\lambda}), \qquad [J_{\mu\nu}, \Theta_\nu]=0,\end{aligned} \tag{4.1}$$

where P_μ, $J_{\nu\mu}$ and Θ_ν are the four vector energy-momentum , the angular momentum tensor and the current density, respectively.

In fact, they obtain from invariance of Lagrangian under some canonical transformation (Poincare transformation) according to Noether's theorem such that one can obtain the constant of motion or show that the conservation laws have the covariant form [16]. For example, from demanding invariance under space-time transformations, the four vector energy-momentum conserved quantity are obtained by

$$P_\mu = \frac{i}{c}\int T_{\mu\nu} ds_\nu, \tag{4.2}$$

where ds_ν is the element of the surface. $T_{\mu\nu}$ is also the well known canonical energy-momentum tensor

$$T_{\mu\nu} = \frac{\partial\mathcal{L}}{\partial\psi_{,\nu}}\psi_{,\mu} - \delta_{\mu\nu}\mathcal{L}, \tag{4.3}$$

where $\mathcal{L}$ is the Lagrangian density of the system and $\psi_{,\nu}$ is the derivative of the field with respect to ν.

Under Lorentz transformations too, we have the following covariant angular momentum tensor

$$J_{\rho,\varphi} = \int (\rho P_\varphi - \varphi P_\rho) ds_\rho d\varphi, \tag{4.4}$$

where its complete form has a spin term which is zero in our problem.

Finally, under phase transformation, we have

$$\Theta_\nu = -iq\frac{\partial\mathcal{L}}{\partial\psi_{,\nu}}\psi, \tag{4.5}$$

where is called the current density for a constant parameter q.

Now, we consider the following Hamiltonian with special interaction between three-body

$$H(x_1, x_2, x_3) = -\sum_{i=1}^{3}\frac{\partial^2}{\partial x_i^2} + \frac{g_1^2}{(x_1-x_2)^2} + \frac{g_2^2}{(x_1+x_2-2x_3)^2}, \tag{4.6}$$

where g_1^2 is the coupling coefficient between the particles 1 and 2 and g_2^2 is the coupling coefficient among three particles. The above Hamiltonian is a special form of the Sutherland potential [9]. If we choose Jacobian coordinate system as

$$x_1 - x_2 = \sqrt{2}u, \qquad x_1 + x_2 - 2x_3 = \sqrt{6}v, \qquad x_1 + x_2 + x_3 = \sqrt{3}X, \tag{4.7}$$

then (4.6) will become as $H(x_1, x_2, x_3) = \tilde{H}(u, v) - \frac{\partial^2}{\partial X^2}$, that is, the center of mass is separated and the spectrum of system is shifted by a constant. Now if we use the polar coordinates as: $u = \rho \sin(\varphi)$, $v = \rho \cos(\varphi)$, and choosing $g_1^2 = 2(m_1^2 - lm_1 + \frac{l^2}{4} - \frac{1}{4})$ and $g_2^2 = 6(m_2^2 - lm_2 + \frac{l^2}{4} - \frac{1}{4})$, then the above Hamiltonian becomes the same contracted Hamiltonian given by (3.6), that is

$$\tilde{H}^c(\rho, \varphi) = P_\rho^2 + P_\varphi^2 + \frac{g_1^2}{2\rho^2 \sin^2(\varphi)} + \frac{g_2^2}{6\rho^2 \cos^2(\varphi)}, \tag{4.8}$$

where $P_\rho^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}$ and $P_\varphi^2 = \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}$.

Now, for getting the conserved quantities of this model, we should compute the Lagrangian of the system. To find the Lagrangian of our Sutherland-like model, we use the Hamiltonian-Jacobi (H-J) method and arrive to the field equations concluding conservative quantities and Poincare Lie algebra.

According to [17], the (H-J) equations take the following form

$$\begin{aligned} &\left(\frac{\partial S_1}{\partial\varphi}\right)^2 + \frac{g_1^2}{2\sin^2(\varphi)} + \frac{g_2^2}{6\cos^2(\varphi)} = \alpha_1^2, \\ &\left(\frac{\partial S_2}{\partial\rho}\right)^2 + \frac{\alpha_1^2}{\rho^2} = E, \end{aligned} \tag{4.9}$$

where α_1 and E are the positive separation constants. After some algebraic calculations and using maple software, one could have the solution of these equations, hence, the total action S is obtained by

$$S = S_1\varphi + S_2\rho - Et,$$

where its wave function is

$$\psi = \psi_0 e^{iS} = \psi_0 e^{i(S_1\varphi + S_2\rho - Et)}.$$

Also, the derivatives of this field are

$$\begin{aligned} \frac{\partial\psi}{\partial\rho} &= i\psi\frac{\partial S}{\partial\rho} = i\psi_0 e^{iS}\sqrt{E - \frac{\alpha_1^2}{\rho^2}} \equiv \psi_{,\rho}, \\ \frac{\partial\psi}{\partial\varphi} &= i\psi\frac{\partial S}{\partial\varphi} = i\psi_0 e^{iS}\sqrt{\alpha_1^2 - \frac{g_1^2}{2\sin^2(\varphi)} - \frac{g_2^2}{6\cos^2(\varphi)}} \equiv \psi_{,\varphi}, \\ \frac{\partial\psi}{\partial t} &= i\psi\frac{\partial S}{\partial t} = i\psi_0 e^{iS}(-E) \equiv \dot{\psi}. \end{aligned} \tag{4.10}$$

The Lagrangian of system is also obtained by

$$L = -E + \frac{\partial S}{\partial q_i}\dot{q}_k,$$

so we have

$$L = -E + \frac{\partial S}{\partial \varphi}\dot{\varphi} + \frac{\partial S}{\partial \rho}\dot{\rho} = \frac{\dot{\psi}}{i\psi} + \frac{\dot{\varphi}}{i\psi}\psi_{,\varphi} + \frac{\dot{\rho}}{i\psi}\psi_{,\rho}. \tag{4.11}$$

Therefore we can compute three conserved quantities: P_μ , $J_{\nu\mu}$ and Θ_ν. For $T_{\mu\nu}$ we have

$$\begin{aligned} T_{\rho\rho} &= -\frac{\dot{\psi}}{i\psi} - \frac{\dot{\varphi}}{i\psi}\psi_{,\varphi}, \qquad T_{\rho\varphi} = \frac{\dot{\varphi}}{i\psi}\psi_{,\rho}, \\ T_{\varphi\rho} &= \frac{\dot{\rho}}{i\psi}\psi_{,\varphi}, \qquad T_{\varphi\varphi} = -\frac{\dot{\psi}}{i\psi} - \frac{\dot{\rho}}{i\psi}\psi_{,\rho}. \end{aligned} \tag{4.12}$$

Also for $J_{\nu\mu}$ we obtain

$$J_{\rho\rho} = 0, \qquad J_{\rho\varphi} = -J_{\varphi\rho} = \rho P_\varphi - \varphi P_\rho. \tag{4.13}$$

Finally for Θ_ν

$$\Theta = -iq\left(\frac{\partial \mathcal{L}}{\partial \psi_{,\rho}}\psi + \frac{\partial \mathcal{L}}{\partial \psi_{,\varphi}}\psi\right) = -q(\dot{\rho} + \dot{\varphi}). \tag{4.14}$$

Now the Poincare Lie algebra is confirmed as follows:

$$\begin{aligned}
[P_\rho, P_\varphi] &= \left[\frac{i}{c}\int (T_{\rho\rho} ds_\rho + T_{\rho\varphi} ds_\varphi), \frac{i}{c}\int (T_{\varphi\varphi} ds_\varphi + T_{\varphi\rho} ds_\rho)\right] \\
&= -\frac{1}{c^2}\Bigg(\int \left[\left(-\frac{\dot{\psi}}{i\psi} - \frac{\dot{\varphi}}{i\psi}\psi_{,\varphi}\right), \frac{\dot{\rho}}{i\psi}\psi_{,\varphi}\right] ds_\rho \\
&\quad + \int \left[\frac{\dot{\varphi}}{i\psi}\psi_{,\rho}, \left(-\frac{\dot{\psi}}{i\psi} - \frac{\dot{\rho}}{i\psi}\psi_{,\rho}\right)\right] ds_\varphi\Bigg) \\
&= 0, \\
[P_\rho, J_{\rho\varphi}] &= \left[P_\rho, \int (\rho P_\varphi - \varphi P_\rho) ds_\rho d\varphi\right] = -iP_\varphi, \\
[P_\rho, \Theta] &= \left[\frac{i}{c}\int (T_{\rho\rho} ds_\rho + T_{\rho\varphi} ds_\varphi), -q(\dot{\rho} + \dot{\varphi})\right] \\
&= \left[\left(\frac{1}{c}\int \left(-\frac{\dot{\psi}}{i\psi} - \frac{\dot{\varphi}}{i\psi}\psi_{,\varphi}\right) ds_\rho + \frac{1}{c}\int \left(\frac{\dot{\varphi}}{i\psi}\psi_{,\rho} ds_\varphi\right)\right), -q(\dot{\rho} + \dot{\varphi})\right] = 0, \\
[P_\varphi, \Theta] &= \left[\frac{i}{c}\int (T_{\rho\rho} ds_\rho + T_{\rho\varphi} ds_\varphi), -q(\dot{\rho} + \dot{\varphi})\right] \\
&= \left[\left(\frac{1}{c}\int \left(-\frac{\dot{\psi}}{i\psi} - \frac{\dot{\varphi}}{i\psi}\psi_{,\varphi}\right) ds_\varphi + \frac{\dot{\rho}}{i\psi}\psi_{,\varphi} ds_\rho\right), -q(\dot{\rho} + \dot{\varphi})\right] = 0, \\
[J_{\rho\varphi}, \Theta] &= \left[\int (\rho P_\varphi - \varphi P_\rho) ds_\rho d\varphi, -q(\dot{\rho} + \dot{\varphi})\right] = 0, \\
[J_{\rho\varphi}, J_{\varphi\rho}] &= 0.
\end{aligned}$$

So three body quantum system has Poincare Lie algebra. Also this system has two variables ρ and φ and three conserved quantities, that commute with the energy operator $H = P^0$. So this system is a superintegrable model too, and it can be solved in different coordinates.

5 Conclusion

In this work, we have shown that by using the Inonu-Wigner contraction, the contracted Hamiltonian on symmetric space $SU(3)/SU(2)$ is a superintegrable Hamiltonian on R^2. We have solved the separated (H-J) equations and have shown that the conservative quantities satisfy the Poincare Lie algebra.

Physical applications deduced from this work are the integrability of the considered system. It makes the possibility of calculating eigenvalues and eigenfunctions of the quantum system, as well as trajectories in the classical ones. It also seems that by choosing another parameterization of the $SU(3)$ group or another reduction method such as Marsden-Weinstein one can find other quantum integrable system.

References

1. Grabowski, J., Landi, G., Marmo, G., Vilasi, G.: Generalized reduction procedure: symplectic and poisson formalism. Fortschr. Phys. **42**, 393–427 (1994)
2. Morse, P.M.: Diatomic molecules according to the wave mechanics. II. Vibrational levels. Phys. Rev. **34**, 57 (1929)
3. Poschl, G., Teller, E.: Bemerkungen zur Quantenmechanik des Anharmonischen Oszillators. Z. Phys. **83**, 143 (1933)
4. Winternitz, P., Smorodinsky, Ya.A., Uhlir, M., Frits, I.: Symmetry groups in classical and quantum mechanics. Sov. J. Nucl. Phys. **4**, 444 (1967)
5. Evans, N.W.: Superintegrability in classical mechanics. Phys. Rev. A **41**, 5666 (1990)
6. Evans, N.W.: Super-integrability of the Winternitz system. Phys. Lett. A **147**, 483 (1990)
7. Evans, N.W.: Group theory of the Smorodinsky-Winternitz system. J. Math. Phys. **32**, 3369 (1991)
8. Calogero, F.: Solution of a three-body problem in one dimension. J. Math. Phys. **10**, 2191 (1969)
9. Sutherland, B.: Exact results for a quantum many-body problem in one dimension. Phys. Rev. A **4**, 2019 (1971)
10. Calzada, J.A., Del Olmo, M.A., Rodrguez, M.A.: Pseudo-orthogonal groups and integrable dynamical systems in two dimensions. J. Math. Phys. **40**, 188 (1999)
11. Delolmo, M.A., Calzada, J.A., Rodrguez, M.A.: Classical superintegrable $SO(p,q)$ Hamiltonian systems. J. Geom. Phys. **23**, 14 (1997)
12. Izmest'ev, A.A., Pogosyan, G.S., Sissakian, A.N., Winternitz, P.: Contractions of Lie algebras and separation of variables. J. Phys. A, Math. Gen. **29**, 5949 (1996)
13. Panahi, H., Jafarizadeh, M.A.: Two and three dimensional Hamiltonians with generalized and ordinary shape invariance symmetry. J. Math. Phys. **46**, 012103 (2005)
14. Rowe, D.J., Sanders, B.C., De Guise, H.: Representations of the Weyl group and Wigner functions for $SU(3)$. J. Math. Phys. **40**, 3604 (1999)
15. Gilmore, R.: Lie Groups, Lie Algebra and Some of Their Applications. Wiley, New York (2006)
16. Roman, P.: Introduction to Quantum Field Theory. Wiley, New York (1969)
17. Goldstein, H., Poole, C., Safko, J.: Classical Mechanics. Addison Wesley, Reading (2000)